Signals and Systems

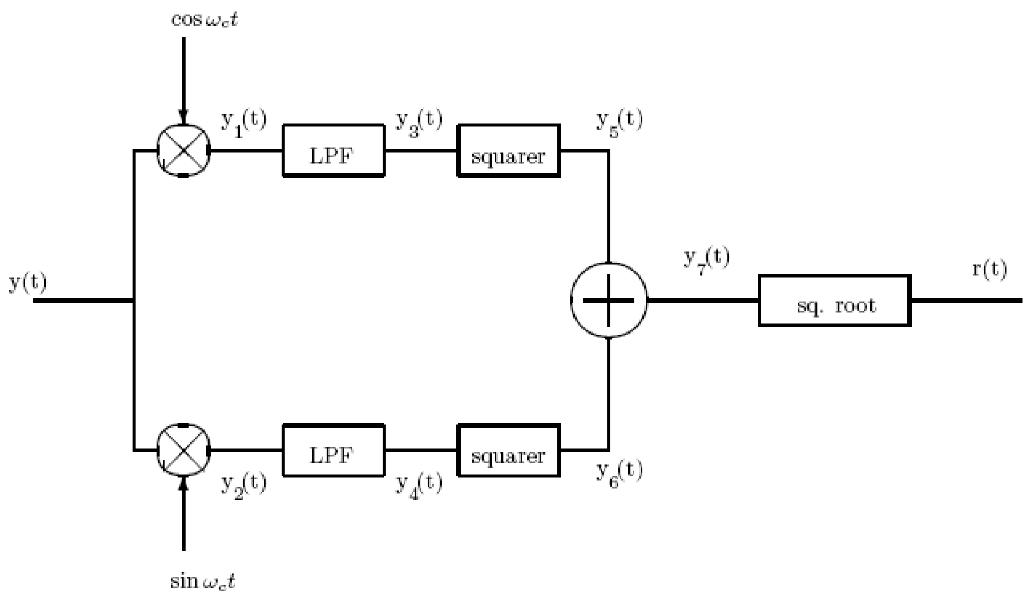
Solutions to Homework Assignment # 10

Problem 1. (O&W Problem 8.26)

The system in question illustrate show some of the basic trigonometric identities have very interesting consequences when creatively applied to a communication system. We will start our analysis at the system input, where we have a constant but unknown phase Θ_c and an input of

$$y(t) = [x(t) + A]\cos(\omega_c t + \Theta_c)$$

Denote the intermediate signals in the system as shown below.



And now we crunch through the algebra...

$$y_1(t) = [x(t) + A]\cos(\omega_c t + \Theta_c)\cos(\omega_c t)$$

$$y_1(t) = [x(t) + A][\ \tfrac{1}{2}\cos\Theta_c + \tfrac{1}{2}\cos(2\omega_c t + \Theta_c)]$$

$$y_1(t) = [x(t) + A] \times \tfrac{1}{2}\cos\Theta_c + [x(t) + A] \times \cos(2\omega_c t + \Theta_c)$$

Since our lowpass filter removes all spectral components with radial frequency $\omega > \omega_c$, the second term in the final expression for $y_1(t)$ will be removed. This leaves us with (after lowpass filtering)

$$y_3(t) = [x(t) + A] \times \tfrac{1}{2}\cos\Theta_c$$

And gives the immediate result of

$$y_5(t) = [x(t) + A]^2 \times \tfrac{1}{4}\cos^2\Theta c$$

We proceed in a very similar manner for the lower half of our system, except we find that the first term in the expression for $y_2(t)$ is now removed by the LPF.

$$y_2(t) = [x(t) + A] \times \tfrac{1}{2}\sin(2\omega_c t + \Theta_c) - [x(t) + A] \times \tfrac{1}{2}\sin\Theta_c$$

$$y_4(t) = -[x(t) + A] \times \tfrac{1}{2} \sin\Theta_c$$

$$y_6(t) = [x(t) + A]^2 \times \tfrac{1}{4} \sin^2 \Theta_c$$

The rest of the system is quite straightforward, as we have

$$y_7(t) = \tfrac{1}{4} [x(t) + A]^2 \times (\cos^2 \Theta_c + \sin^2 \Theta_c)$$

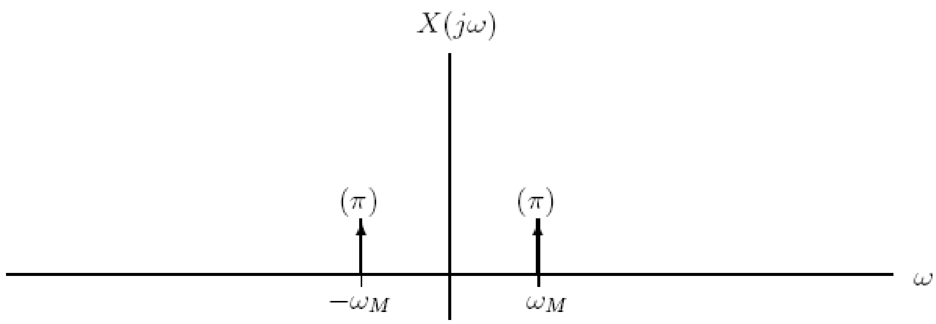
$$y_7(t) = \tfrac{1}{4} [x(t) + A]^2$$

$$r(t) = \tfrac{1}{2} [x(t) + A]$$

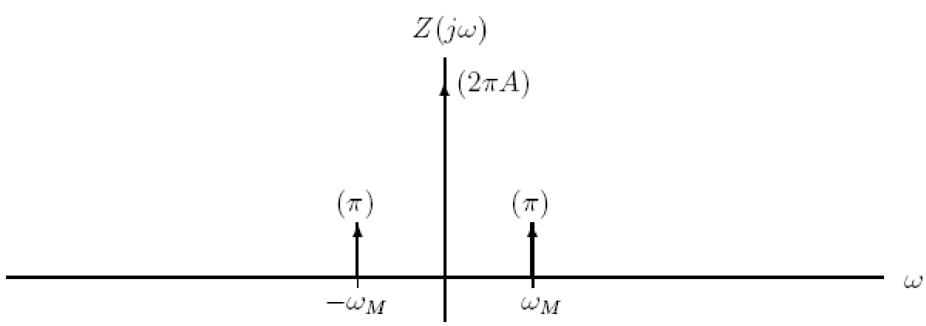
From which $x(t)$ can be recovered by amplification by a factor of 2 and then using normal envelope detection.

Problem 2. O&W 8.27

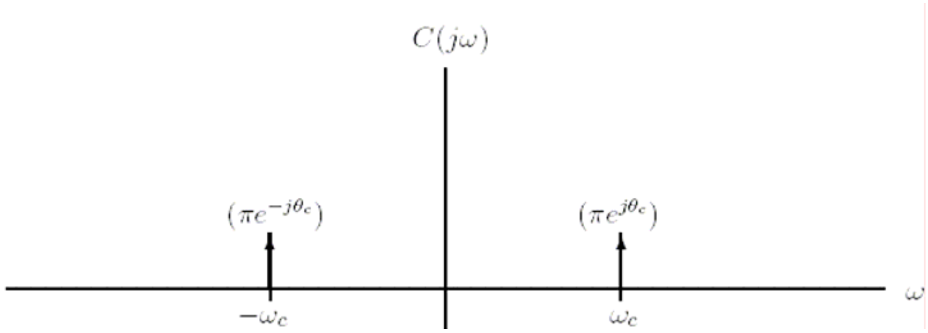
(a) We have $x(t) = \cos\omega_M t$, so $X(j\omega) = \pi(\delta(\omega-\omega_M) + \delta(\omega+\omega_M))$:



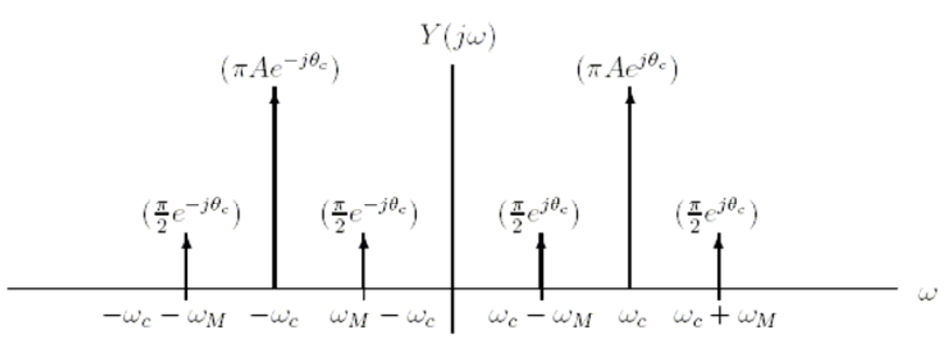
Let $z(t) = A + x(t)$. Then, $Z(j\omega) = X(j\omega) + 2\pi A\delta(\omega)$:



Let $c(t) = \cos(\omega_c t+\theta_c)$. Thus, $C(j\omega) = \pi(e^{j\theta_c} \delta(\omega-\omega_M) + e^{-j\theta_c} \delta(\omega+\omega_M))$:



We have $y(t) = z(t)c(t)$, so $Y(j\omega) = (\frac{1}{2\pi})\, Z(j\omega) * C(j\omega)$:



Since $y(t)$ is periodic, we can express it in terms of its Fourier series coefficients a_k. We know $y(t)$ is real, so $y(t) = |y(t)|$. Thus, by Parseval's relation,

$$\begin{aligned} P_y &= \frac{1}{T}\int_T y^2(t)dt \\ &= \frac{1}{T}\int_T |y(t)|^2 dt \\ &= \sum_{k=-\infty}^{+\infty} |a_k|^2 \end{aligned}$$

The formula for translating the Fourier series coefficients a_k of a periodic signal $y(t)$ into its Fourier transform $Y(j\omega)$ is

$$Y(j\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0),$$

where ω_0 is the fundamental frequency. Hence, we can find the Fourier series coefficients a_k by taking the area of each impulse and dividing by 2π.

(Sidenote: $y(t)$ is periodic if and only if the frequencies at which the impulses occur in the Fourier transform are harmonically related. In other words, there must exist a real number ω_0 and integers k_c and k_M such that

$$\omega_c = k_c\omega_0$$

$$\omega_M = k_M\omega_0.$$

This is the condition we need to discretize the frequency axis from the Fourier transform to the Fourier series representations. However, the problem assumed that $y(t)$ was periodic, so we can convert its Fourier transform to the Fourier series coefficients as done above.)

Thus, the contribution of each sideband impulse to the power is

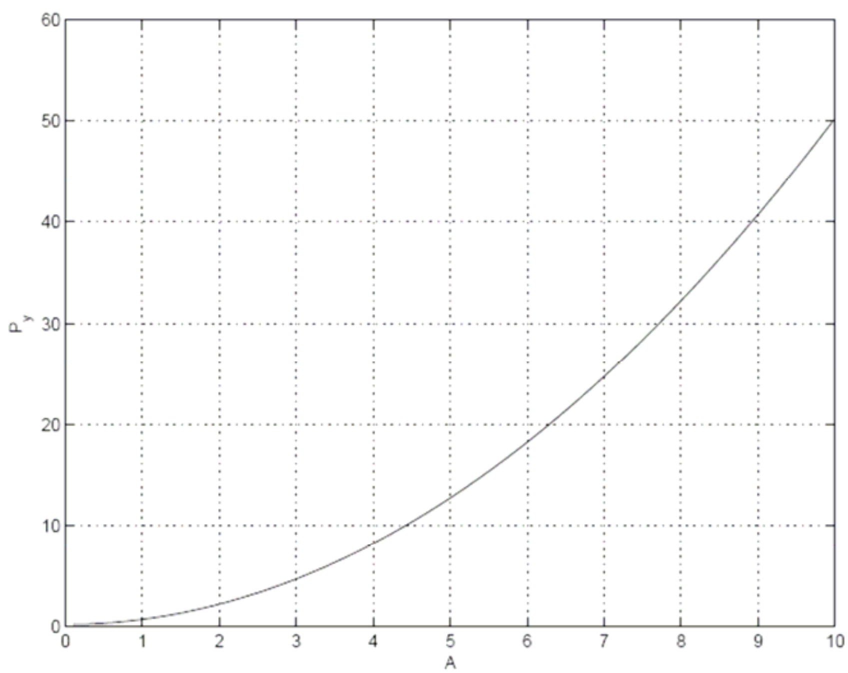
$$\begin{aligned} |a_{\text{sideband}}|^2 &= \left| \frac{\frac{\pi}{2} e^{\pm j\theta_c}}{2\pi} \right|^2 \\ &= \left(\frac{1}{4}\right)^2 |e^{\pm j\theta_c}|^2 \\ &= \frac{1}{16}. \end{aligned}$$

Likewise, the contribution of each "A" impulse to the power is

$$\begin{aligned} |a_A|^2 &= \left| \frac{\pi A e^{\pm j\theta_c}}{2\pi} \right|^2 \\ &= \left(\frac{A}{2}\right)^2 |e^{\pm j\theta_c}|^2 \\ &= \frac{A^2}{4}. \end{aligned}$$

So,

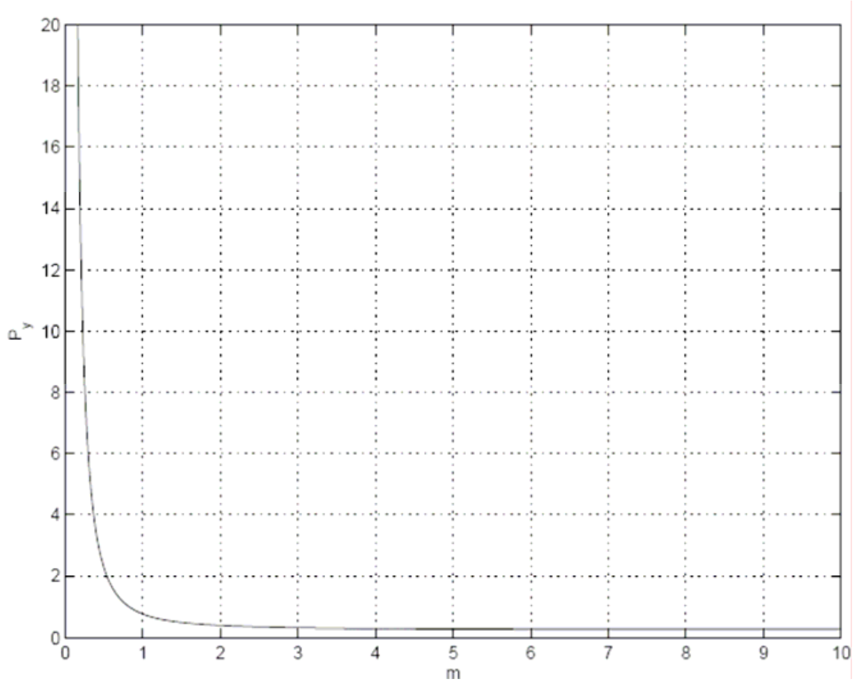
$$\begin{aligned} P_y &= 4\left(\frac{1}{16}\right) + 2\left(\frac{A^2}{4}\right) \\ \Longrightarrow \quad P_y &= \frac{1}{4} + \frac{A^2}{2} \end{aligned} \tag{1}$$



The maximum value of $|x(t)|$ is 1, so the modulation index is $m = 1/A$.

Plugging $m = 1/A \Rightarrow A = 1/m$ into Eq.1 produces

$$P_y = \frac{1}{4} + \frac{1}{2m^2}$$



(b) The effciency is

$$\begin{aligned} \epsilon &= \frac{\text{power in sidebands}}{\text{total power}} \\ &= \frac{4\left(\frac{1}{16}\right)}{4\left(\frac{1}{16}\right) + 2\left(\frac{A^2}{4}\right)} \\ \Longrightarrow \quad \epsilon &= \frac{1}{1 + 2A^2}. \qquad (2) \end{aligned}$$

Substituting in $A = 1/m$ into Eq.2 produces

$$\epsilon = \frac{m^2}{m^2 + 2}.$$

The MATLAB code that produced the graphs in parts b and d is:

```
% part a
% graph P_y as function of A and m
m = [.1:.001:10]; A = 1./m; P_y = 1/4 + 1./(2*m.^2); close all;
figure; plot(A, P_y); grid on; xlabel('A'); ylabel('P_{y}');
figure; plot(m, P_y); axis([0, 10, 0, 20]); grid on; xlabel('m');
ylabel('P_{y}');

% part b
```

```
% graph efficiency as function of m
eff = (m.^2)./(m.^2 + 2); figure; plot(m, eff); grid on;
xlabel('modulation index m'); ylabel('efficiency \epsilon');
```

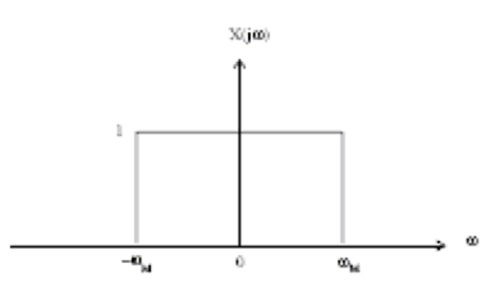
Problem 3. (O&W 8.34)

Looking at Figure P8.34 on page 640 of O&W, we'll name a new variable, $z(t)$, that is the signal between the square-law device and bandpass filter, $H(j\omega)$.

$$z(t) = (x(t) + \cos(\omega_c t))^2 = x^2(t) + 2x(t)\cos(\omega_c t) + \cos^2(\omega_c t)$$

$$Z(j\omega) = (1/2\pi)X(j\omega)*X(j\omega) + [X(j(\omega+\omega_c)) + X(j(\omega-\omega_c))]$$
$$+ 1/2[\delta(\omega+2\omega_c) + \delta(\omega-2\omega_c)] + \delta(\omega)$$

Since $X(j\omega)$ is band limited, $X(j\omega) = 0$ $|\omega| > \omega_M$, the convolution of $X(j\omega)$ with itself is also band limited ($X(j\omega) = X(j\omega) = 0$ $|\omega| > 2\omega_M$). With that said, we are safe to pull out $x(t)\cos(\omega_c t)$ with a bandpass filter given in $H(j\omega)$. Consider an arbitrary function $X(j\omega)$ sketched below:



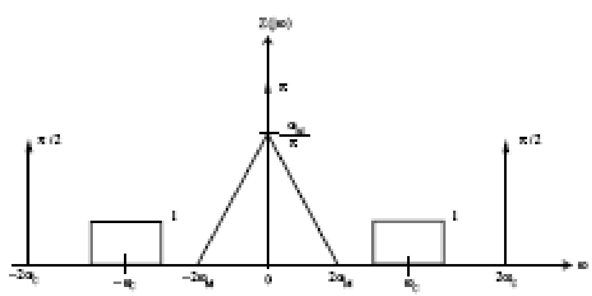
and the spectrum of the output of the square-law device, $Z(j\omega)$, is sketched below. To make sure that $H(j\omega)$ doesn't have any of the signal from $X(j\omega)*X(j\omega)$ in the passband, we have $2\omega_M < \omega_c-\omega_M$. Therefore, we can see the range of values for ω_l and ω_h in the sketch below.

$$2\omega_M < \omega_l < \omega_c - \omega_M$$

$$\omega_c + \omega_M < \omega_h < 2\omega_c$$

Our goal is to have $x(t)\cos(\omega_c t)$ come out of the bandpass filter; however, the amplitude is off by a factor of ½ (see figure below), thus A has to be equal to ½, since the Fourier transform of $x(t)\cos(\omega_c t)$ is:

$$FT\{x(t)\cos(\omega_c t)\} = ½\,[X(j(\omega+\omega_c)) + X(j(\omega-\omega_c))]$$



Problem 4. O&W Problem 8.47

This problem illustrates the detrimental effects of losing phase synchronization between the transmitter and receiver in a digital system. To gain insight for this problem, let's look at the following:

$$y[n] = x[n]\cos[\omega_c n + \Theta_c]$$

$$w[n] = y[n]\cos[\omega_d n + \Theta_d]$$

$$= x[n]\cos[\omega_c n + \Theta_c]\cos[\omega_d n + \Theta_d]$$

One way to solve this problem is using Euler's relation (note that you can also do this using trigonometric identities as well):

$$= \quad x[n]\frac{1}{2}\left(e^{j\omega_c n}e^{j\Theta_c} + e^{-j\omega_c n}e^{-j\Theta_c}\right)\frac{1}{2}\left(e^{j\omega_d n}e^{j\Theta_d} + e^{-j\omega_d n}e^{j\Theta_d}\right)$$

Which simplifies to:

$$\begin{aligned} w[n] \quad = \quad x[n] \quad & \tfrac{1}{4}[e^{j\omega_c n}e^{j\Theta_c}e^{j\omega_d n}e^{j\Theta_d} \quad + \quad e^{j\omega_c n}e^{j\Theta_c}e^{-j\omega_d n}e^{-j\Theta_d} \\ & +e^{-j\omega_c n}e^{-j\Theta_c}e^{j\omega_d n}e^{j\Theta_d} \quad + \quad e^{-j\omega_c n}e^{-j\Theta_c}e^{-j\omega_d n}e^{-j\Theta_d}] \end{aligned}$$

Notice that by combining these exponential terms, we get:

$$\begin{aligned} w[n] \quad = \quad x[n] \quad & \tfrac{1}{4}[e^{j((\omega_c+\omega_d)n+(\Theta_c+\Theta_d))} \quad + \quad e^{j((\omega_c-\omega_d)n+(\Theta_c-\Theta_d))} \\ & +e^{-j((\omega_c-\omega_d)n+(\Theta_c-\Theta_d))} \quad + \quad e^{-j((\omega_c+\omega_d)n+(\Theta_c+\Theta_d))}] \end{aligned}$$

Which simplifies to:

$$w[n] \quad = \quad x[n]\left[\frac{1}{2}\cos[(\omega_c+\omega_d)n + (\Theta_c+\Theta_d)] + \frac{1}{2}\cos[(\omega_c-\omega_d)n + (\Theta_c-\Theta_d)]\right]$$

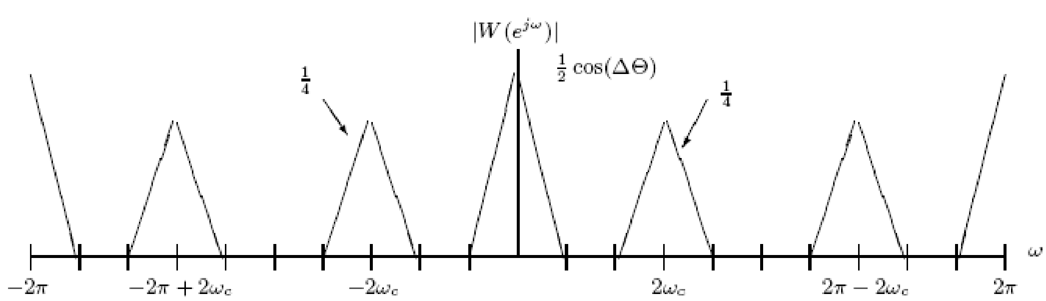
$$w[n] \quad = \quad x[n]\left[\frac{1}{2}\cos[(\omega_c+\omega_d)n + (\Theta_c+\Theta_d)] + \frac{1}{2}\cos[\Delta\omega n + \Delta\Theta]\right]$$

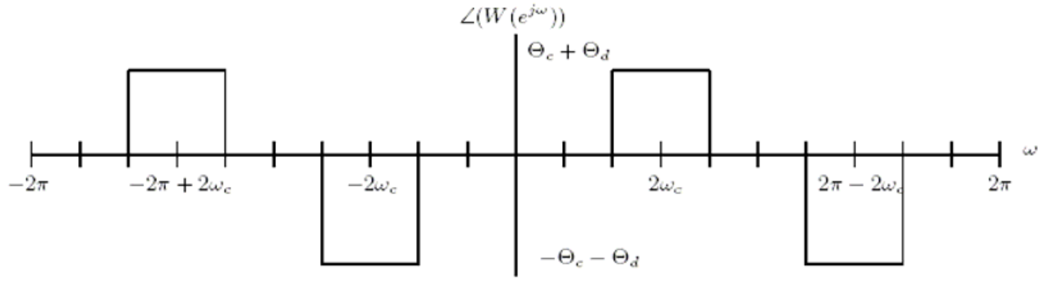
where $\Delta\omega = \omega_c - \omega_d$ and $\Delta\Theta = \Theta_c - \Theta_d$. Notice that by doing this algebra first, we can use it for the rest of the problem.

(a) If we assume $\Delta\omega = 0$, then $\omega_c = \omega_d$ and we can simplify $w[n]$ to the following:

$$w[n] = x[n][\ \tfrac{1}{2}\cos[2\omega_c n + (\Theta_c + \Theta_d)] + \tfrac{1}{2}\cos[\Delta\Theta]]$$

From this equation, we can plot the magnitude and phase of $W(e^{j\omega})$:





(b) Again, if $\Delta\omega = 0$, then $w[n]$ will be the same as in part (a). If we pick $w = \omega_M$, then:

$$r[n] = x[n]\cos[\Delta\Theta]$$

But, notice that if $\Delta\Theta = \pi/2$, then since $\cos(\pi/2) = 0$, then $r[n] = 0$! This is a very interesting result! We see that a phase offset between the transmitter and receiver creates an interferometric effect at the receiver output, with the amplitude of $r[n]$ depending directly on the phase mismatch $\Delta\Theta$. Clearly this is not desirable, which leads to the requirement for a complex receiver to maintain phase coherence in such a system.

(c) Here we have the situation where there is no phase difference between the transmitter and receiver but there is instead a frequency offset $\Delta\omega$. If $\Delta\Theta = 0$, then $\Theta_c = \Theta_d$ and $w[n]$ will simplify to the following:

$$w[n] = x[n][\ ½ \cos[(\omega_c+\omega_d)n + 2\Theta_c] + ½ \cos[\Delta\omega n]]$$

If we let $w = \omega_M + \Delta\omega$, then:

$$r[n] = x[n]\cos[\Delta\omega n]$$

We see the same interesting result here as in part (b), except this time the interferometric relationship is governed by $\Delta\omega n$ instead of $\Delta\Theta$.

Problem 5. O&W Problem 8.48

(a) Since $p[n]$ is periodic, we can use the DT Fourier Series analysis equation to find the general form for a_k and then use the relationship that

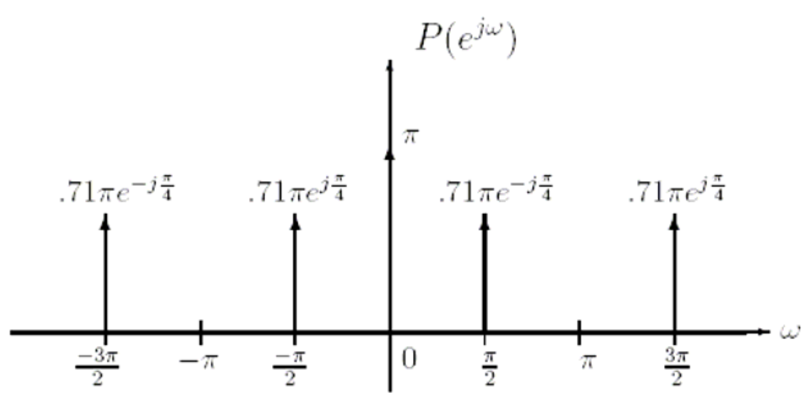
$$P(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - \tfrac{2\pi k}{N}):$$

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=<N>} p[n]e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=0}^{M} e^{-jk\frac{2\pi}{N}n} \\
a_k &= \frac{1}{N}\frac{1-e^{-jk\frac{2\pi}{N}(M+1)}}{1-e^{-jk\frac{2\pi}{N}}} \qquad \text{since:} \quad \sum_{l=0}^{B} a^l = \frac{1-a^{B+1}}{1-a} \\
&= \frac{1}{N}\frac{e^{-jk\frac{2\pi}{N}(\frac{M+1}{2})}\left(e^{jk\frac{2\pi}{N}(\frac{M+1}{2})}-e^{-jk\frac{2\pi}{N}(\frac{M+1}{2})}\right)}{e^{-jk\frac{2\pi}{N}(\frac{1}{2})}\left(e^{jk\frac{2\pi}{N}(\frac{1}{2})}-e^{-jk\frac{2\pi}{N}(\frac{1}{2})}\right)} \\
&= \frac{1}{N}e^{-jk\frac{2\pi}{N}(\frac{M}{2})}\left[\frac{2j\sin(\frac{k\pi}{N}(M+1))}{2j\sin(\frac{k\pi}{N})}\right] \\
&= e^{-j\frac{kM\pi}{N}}\left[\frac{\sin(\frac{k\pi}{N}(M+1))}{N\sin(\frac{k\pi}{N})}\right]
\end{aligned}
$$

Note: You can also use Tables 5.1 and 5.2 on pp391-392 to find this result. Therefore:

$$
\begin{aligned}
P(e^{j\omega}) &= 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta(\omega-\frac{2\pi k}{N}) \qquad \text{where} \\
a_k &= \begin{cases} e^{-j\frac{kM\pi}{N}}\left[\frac{\sin(\frac{k\pi}{N}(M+1))}{N\sin(\frac{k\pi}{N})}\right] & , \quad k \neq Ni \text{ for } i \in \mathbb{Z} \\ \frac{M+1}{N} & , \quad k = Ni \text{ for } i \in \mathbb{Z} \end{cases}
\end{aligned}
$$

For $N = 4$, $M = 1$, $P(e^{j\omega})$ will look like:



(b) Let $\omega_M = \pi/2N$. Since $M = 1$, then:

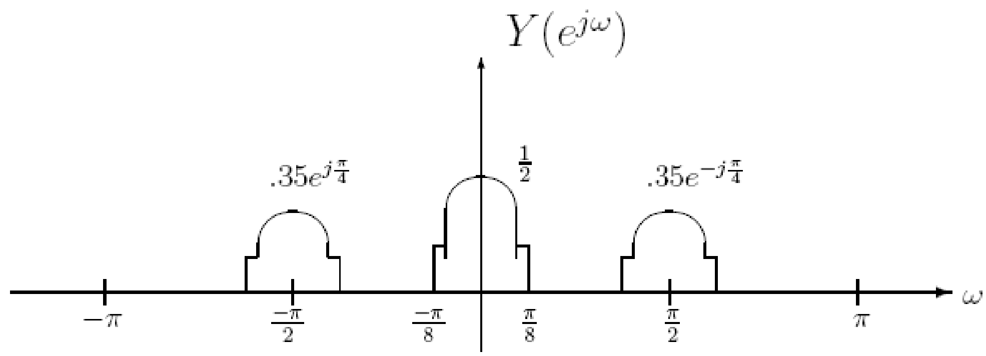
$$P(e^{j\omega}) \;=\; 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \qquad \text{where}$$

$$a_k \;=\; \begin{cases} e^{-j\frac{k\pi}{N}} \left[\frac{\sin(\frac{2\pi k}{N})}{N\sin(\frac{k\pi}{N})}\right] & , \quad k \neq Ni \text{ for } i \in \mathbb{Z} \\ \frac{2}{N} & , \quad k = Ni \text{ for } i \in \mathbb{Z} \end{cases}$$

Since $2.\pi/2N = \pi/N < 2\pi/N$ (no aliasing) and $Y(e^{j\omega}) = 1/2\pi X(e^{j\omega}) * P(e^{j\omega})$, then:

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} a_k X(e^{j(\omega - \frac{2\pi k}{N})})$$

For $N = 4$, $Y(e^{j\omega})$ will look like:



(c) For this problem, we are given:

$$X(e^{j\omega}) = \begin{cases} \neq 0 & , \quad |\omega| < \omega_M \\ 0 & , \quad \omega_M < |\omega| \leq \pi \end{cases}$$

where $X(e^{j\omega})$ repeats every 2π. In order to make sure that there is no aliasing:

$$2\omega_M < \omega_s = 2\pi/N$$

Therefore, $\omega_M < \pi/N$ where $N \in Z^+$. Notice that this calculation does not depend on M.

(d) To get back the original signal, we need to low-pass filter $y[n]$: